

## Limit of a function ( examples - II part )

In following tasks we will use “the known” limit:

$$\boxed{\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1} \quad \text{More can be used:} \quad \lim_{x \rightarrow 0} \frac{\sin ax}{ax} = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{\sin^n ax}{(ax)^n} = 1$$

1) Find the following limits:

a)  $\lim_{x \rightarrow 0} \frac{\sin 4x}{x};$

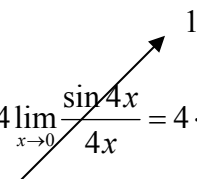
b)  $\lim_{x \rightarrow 0} \frac{\operatorname{tg} x}{x};$

c)  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2};$

d)  $\lim_{x \rightarrow a} \frac{\sin x - \sin a}{x - a};$

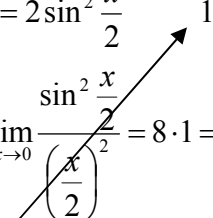
**Solution:**

a)  $\lim_{x \rightarrow 0} \frac{\sin 4x}{x};$  [up and down, add 4] =  $\lim_{x \rightarrow 0} \frac{4 \sin 4x}{4x} = 4 \lim_{x \rightarrow 0} \frac{\sin 4x}{4x} = 4 \cdot 1 = 4$



b)  $\lim_{x \rightarrow 0} \frac{\operatorname{tg} x}{x} = \lim_{x \rightarrow 0} \frac{\frac{\sin x}{\cos x}}{x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} =$   
 $= 1 \cdot \lim_{x \rightarrow 0} \frac{1}{\cos x} = 1 \cdot \frac{1}{\cos 0} = 1 \cdot \frac{1}{1} = 1$

c)  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} =$  [use formula from trigonometry]:  $1 - \cos x = 2 \sin^2 \frac{x}{2}$   
 $= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x^2} =$  [up and down, add 4] =  $\lim_{x \rightarrow 0} 2 \cdot \frac{4 \sin^2 \frac{x}{2}}{x^2} = 8 \cdot \lim_{x \rightarrow 0} \frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2} = 8 \cdot 1 = 8$



$$\begin{aligned}
 \text{d) } \lim_{x \rightarrow a} \frac{\sin x - \sin a}{x - a} &= \text{use formula from trigonometry } \sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \\
 &= \lim_{x \rightarrow a} \frac{2 \cos \frac{x+a}{2} \sin \frac{x-a}{2}}{x-a} = \\
 &= \lim_{x \rightarrow a} \cos \frac{x+a}{2} \cdot \frac{\sin \frac{x-a}{2}}{\frac{x-a}{2}} = \lim_{x \rightarrow a} \cos \frac{x+a}{2} = \\
 &= \cos \frac{a+a}{2} = \cos \frac{2a}{2} = \cos a
 \end{aligned}$$

2) Calculate the following limits:

$$\text{a) } \lim_{x \rightarrow 0} \frac{\sin 4x}{\sqrt{x+1}-1};$$

$$\text{b) } \lim_{x \rightarrow \pi} \frac{\cos \frac{x}{2}}{x - \pi};$$

$$\text{c) } \lim_{x \rightarrow 1} \frac{\sin(1-x)}{\sqrt{x}-1};$$

**Solution:**

a)

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\sin 4x}{\sqrt{x+1}-1} &= \text{first rationalization} \\
 &= \lim_{x \rightarrow 0} \frac{\sin 4x}{\sqrt{x+1}-1} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \\
 &= \lim_{x \rightarrow 0} \frac{\sin 4x \cdot (\sqrt{x+1}+1)}{x+1-1} = \lim_{x \rightarrow 0} \frac{\sin 4x \cdot (\sqrt{x+1}+1)}{x}
 \end{aligned}$$

[up and down, add 4]

$$\begin{aligned}
 &= \lim_{x \rightarrow 0} \frac{4 \sin 4x (\sqrt{x+1}+1)}{4x} = \lim_{x \rightarrow 0} 4 (\sqrt{x+1}+1) \\
 &4(\sqrt{0+1}+1) = 4 \cdot 2 = 8
 \end{aligned}$$

b)

$$\lim_{x \rightarrow \pi} \frac{\cos \frac{x}{2}}{x - \pi} = \text{replacement: } x - \pi = t, \text{ when } x \rightarrow \pi, \text{ then } t \rightarrow \pi - \pi = 0, \text{ so: } t \rightarrow 0$$

$$\lim_{t \rightarrow 0} \frac{\cos \frac{t + \pi}{2}}{t} = \lim_{t \rightarrow 0} \frac{\cos \left( \frac{\pi}{2} + \frac{t}{2} \right)}{t} = \lim_{t \rightarrow 0} \frac{-\sin \frac{t}{2}}{t} \quad (\text{because: } \cos \left( \frac{\pi}{2} + \alpha \right) = -\sin \alpha)$$

$$\lim_{t \rightarrow 0} -\frac{\sin \frac{t}{2}}{2 \cdot \frac{t}{2}} = -\frac{1}{2} \quad \text{because:} \quad \lim_{t \rightarrow 0} \frac{\sin \frac{t}{2}}{\frac{t}{2}} = 1 \longleftrightarrow \lim_{x \rightarrow 0} \frac{\sin ax}{ax} = 1$$

c)

$$\lim_{x \rightarrow 1} \frac{\sin(1-x)}{\sqrt{x}-1} = \text{first rationalization}$$

$$\lim_{x \rightarrow 1} \frac{\sin(1-x)}{\sqrt{x}-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} = \lim_{x \rightarrow 1} \frac{\sin(1-x)(\sqrt{x}+1)}{x-1} = \text{replacement } x-1=t, \text{ when } x \rightarrow 1$$

then  $t \rightarrow 0$

$$= \lim_{t \rightarrow 0} \frac{\sin(-t)(\sqrt{t+1}+1)}{t} = -\lim_{t \rightarrow 0} (\sqrt{t+1}+1) = -(1+1) = -2$$

In next tasks we will use other “the known” limit:

$$\boxed{\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e}$$

and

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{ax}\right)^{ax} = e$$

And the fact:  $\lim_{x \rightarrow a} e^{f(x)} = e^{\lim_{x \rightarrow a} f(x)}$

**3) Calculate the following limits:**

a)  $\lim_{x \rightarrow \infty} \left(1 + \frac{3}{x}\right)^x$ ;

b)  $\lim_{x \rightarrow \infty} \left(\frac{x+1}{x-1}\right)^x$ ;

c)  $\lim_{x \rightarrow \infty} x(\ln(x+1) - \ln x)$ ;

**Solution:**

a)

$$\lim_{x \rightarrow \infty} \left(1 + \frac{3}{x}\right)^x = \text{here where is 3 must be 1, and we will put 3 below} = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{\frac{x}{3}}\right)^x = \text{next}$$

$$\text{exponent multiply and divide with 3} = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{\frac{x}{3}}\right)^{\frac{x}{3} \cdot 3} = e^3 \quad \text{Why?}$$

$$\text{Because : } \lim_{x \rightarrow \infty} \left(1 + \frac{1}{\frac{x}{3}}\right)^{\frac{x}{3}} = e$$

b)

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(\frac{x+1}{x-1}\right)^x &= [\text{trick: in brackets will add 1 and deduct 1}] = \\ \lim_{x \rightarrow \infty} \left(1 + \frac{x+1}{x-1} - 1\right)^x &= \lim_{x \rightarrow \infty} \left(1 + \frac{x+1-x+1}{x-1}\right)^x \\ &= \lim_{x \rightarrow \infty} \left(1 + \frac{2}{x-1}\right)^x = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{\frac{x-1}{2}}\right)^x = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2} \cdot \frac{2}{x-1} \cdot x} = \\ &= \lim_{x \rightarrow \infty} \left(1 + \frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2} \cdot \frac{2x}{x-1}} = \lim_{x \rightarrow \infty} e^{\frac{2x}{x-1}} = e^{\lim_{x \rightarrow \infty} \frac{2x}{x-1}} = e^2 \end{aligned}$$

c)

$$\begin{aligned} \lim_{x \rightarrow \infty} x(\ln(x+1) - \ln x) &= \lim_{x \rightarrow \infty} x \ln \frac{x+1}{x} = \lim_{x \rightarrow \infty} \ln \left(\frac{x+1}{x}\right)^x = \\ &= \ln \lim_{x \rightarrow \infty} \left(\frac{x}{x} + \frac{1}{x}\right)^x = \ln \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = \ln e = 1 \end{aligned}$$

Here we use the rules (see II years logarithms):  $\ln A - \ln B = \ln \frac{A}{B}$  and  $n \circ \ln A = \ln A^n$