Limit of a function (examples - II part)

In following tasks we will use "the known" limit:

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x}=1}\qquad \text{More can be used:}\qquad \lim_{x\to 0}\frac{\sin ax}{ax}=1 \quad\text{and}\quad \lim_{x\to 0}\frac{\sin^n ax}{(ax)^n}=1$$

1) Find the following limits:

a) $\lim\limits_{x\to 0}\frac{\sin 4x}{x}$;

b) $\lim\limits_{x\to 0}\frac{tgx}{x}$;

c) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}$;

d) $\lim\limits_{x\to a}\frac{\sin x-\sin a}{x-a}$;

Solution:

a) $\lim\limits_{x\to 0}\frac{\sin 4x}{x}$; [up and down, add 4]= $\lim\limits_{x\to 0}\frac{4\sin 4x}{4x}=4\lim\limits_{x\to 0}\underbrace{\frac{\sin 4x}{4x}}_{1}=4\cdot 1=4$

b) $$\lim_{x\to 0}\frac{tgx}{x}=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}}{x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}=$$
$$=1\cdot\lim_{x\to 0}\frac{1}{\cos x}=1\cdot\frac{1}{\cos 0}=1\cdot\frac{1}{1}=1$$

c) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}$ = [use formula from trigonometry]: $1-\cos x=2\sin^2\frac{x}{2}$

$$=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=\text{[up and down, add 4]}=\lim_{x\to 0}2\cdot\frac{4\sin^2\frac{x}{2}}{\frac{x^2}{4}}=8\cdot\lim_{x\to 0}\underbrace{\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}}_{1}=8\cdot 1=8$$

d) $\lim\limits_{x\to a} = \dfrac{\sin x - \sin a}{x-a} =$ use formula from trigonometry $\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

$$= \lim_{x\to a}\frac{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}}{x-a} =$$

$$= \lim_{x\to a}\cos\frac{x+a}{2}\cdot\frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}} = \lim_{x\to a}\cos\frac{x+a}{2} =$$

$$= \cos\frac{a+a}{2} = \cos\frac{2a}{2} = \cos a$$

2) Calculate the following limits:

a) $\lim\limits_{x\to 0}\dfrac{\sin 4x}{\sqrt{x+1}-1}$;

b) $\lim\limits_{x\to \pi}\dfrac{\cos\frac{x}{2}}{x-\pi}$;

c) $\lim\limits_{x\to 1}\dfrac{\sin(1-x)}{\sqrt{x}-1}$;

Solution:

a)

$\lim\limits_{x\to 0}\dfrac{\sin 4x}{\sqrt{x+1}-1} =$ first rationalization

$$= \lim_{x\to 0}\frac{\sin 4x}{\sqrt{x+1}-1}\cdot\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} =$$

$$= \lim_{x\to 0}\frac{\sin 4x\cdot\left(\sqrt{x+1}+1\right)}{x+1-1} = \lim_{x\to 0}\frac{\sin 4x\cdot\left(\sqrt{x+1}+1\right)}{x}$$

[up and down, add 4]

$$= \lim_{x\to 0}\frac{4\sin 4x\left(\sqrt{x+1}+1\right)}{4x} = \lim_{x\to 0}4\left(\sqrt{x+1}+1\right)$$

$$4\left(\sqrt{0+1}+1\right) = 4\cdot 2 = 8$$

b)

$$\lim_{x\to\pi}\frac{\cos\frac{x}{2}}{x-\pi}=\text{replacement: } x-\pi=t,\text{when } x\to\pi\text{, then } t\to\pi-\pi=0,\text{ so: } t\to 0$$

$$\lim_{t\to 0}\frac{\cos\frac{t+\pi}{2}}{t}=\lim_{t\to 0}\frac{\cos\left(\frac{\pi}{2}+\frac{t}{2}\right)}{t}=\lim_{t\to 0}\frac{-\sin\frac{t}{2}}{t}\quad (\text{ because : } \cos\left(\frac{\pi}{2}+\alpha\right)=-\sin\alpha\ )$$

$$\lim_{t\to 0}-\frac{\sin\frac{t}{2}}{2\cdot\frac{t}{2}}=-\frac{1}{2}\qquad \text{because :}\qquad \lim_{t\to 0}\frac{\sin\frac{t}{2}}{\frac{t}{2}}=1\longleftrightarrow\lim_{x\to 0}\frac{\sin ax}{ax}=1$$

c)

$$\lim_{x\to 1}\frac{\sin(1-x)}{\sqrt{x}-1}=\text{ first rationalization}$$

$$\lim_{x\to 1}\frac{\sin(1-x)}{\sqrt{x}-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}=\lim_{x\to 1}\frac{\sin(1-x)(\sqrt{x}+1)}{x-1}=\text{ replacement } x-1=t,\text{ when } x\to 1$$

then $t\to 0$

$$=\lim_{t\to 0}\frac{\sin(-t)\left(\sqrt{t+1}+1\right)}{t}=-\lim_{t\to 0}\left(\sqrt{t+1}+1\right)=-(1+1)=-2$$

In next tasks we will use other “the known” limit:

$$\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e}\qquad\text{and}\qquad\lim_{x\to\infty}\left(1+\frac{1}{ax}\right)^{ax}=e$$

And the fact : $\lim_{x\to a}e^{f(x)}=e^{\lim_{x\to a}f(x)}$

3) Calculate the following limits:

a) $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{x}$;

b) $\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^{x}$;

c) $\lim_{x\to\infty}x(\ln(x+1)-\ln x)$;

Solution:

a)

$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{x}$ = here where is 3 must be 1, and we will put 3 below $=\lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{x}$ = next

exponent multiply and divide with $3=\lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{\frac{x}{3}\cdot 3}=e^{3}$ Why?

Because : $\lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{\frac{x}{3}\cdot}=e$

b)

$\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^{x}$ = [trick: in brackets will add 1 and deduct 1]=

$\lim_{x\to\infty}\left(1+\frac{x+1}{x-1}-1\right)^{x}=\lim_{x\to\infty}\left(1+\frac{x+1-x+1}{x-1}\right)$

$=\lim_{x\to\infty}\left(1+\frac{2}{x-1}\right)^{x}=\lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{x}=\lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}\cdot\frac{2}{x-1}\cdot x}=$

$=\lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}\cdot\frac{2x}{x-1}}=\lim_{x\to\infty}e^{\frac{2x}{x-1}}=e^{\lim_{x\to\infty}\frac{2x}{x-1}}=e^{2}$

c)

$\lim_{x\to\infty}x(\ln(x+1)-\ln x)=\lim_{x\to\infty}x\ln\frac{x+1}{x}=\lim_{x\to\infty}\ln\left(\frac{x+1}{x}\right)^{x}=$

$=\ln\lim_{x\to\infty}\left(\frac{x}{x}+\frac{1}{x}\right)^{x}=\ln\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=\ln e=1$

Here we use the rules (see II years logarithms): lnA - lnB = $\ln\frac{A}{B}$ and $n\circ\ln A=\ln A^{n}$